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ON THE PLANE CONTACT PROBLEM FOR A FRICTIONLESS ELASTIC LAYER†

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Abstract—The plane contact problem for an elastic layer lying on an elastic half space is considered. A compressive load is applied to the layer through a frictionless rigid stamp. It is assumed that the contact between the layer and the subspace is also frictionless and only compressive normal tractions can be transmitted through the interface. Hence, the width of the contact area along the layer-subspace interface is finite and is unknown. The problem is formulated in terms of a system of singular integral equations in which the unknown functions are the pressure between the stamp and the layer and that between the layer and the subspace. First the problem is solved for the special case of concentrated loads. Two typical stamp geometries, namely, a flat stamp with sharp corners and a curved stamp, are then investigated. In the case of flat stamp the system of integral equations is homogeneous and as a consequence the width of the contact area between the layer and the subspace turns out to be independent of the magnitude of the external load applied to the stamp. In the curved stamp problem, however, the size of this contact region is a function of the magnitude of the external load.

1. INTRODUCTION

IN THIS paper we will reconsider the plane contact problem for an elastic layer lying on an elastic half space. Due to its application to a variety of important structural problems, in the past the problem has attracted considerable attention (see, for example, [1–3] for the general description of the contact problems in elastic solids, and [4–12] for some recent solutions regarding the problem of the elastic layer). In the great majority of the papers published on the subject, it is assumed that both the stamp and the subspace are rigid. However, at the expense of some increase in the analysis and in the numerical work, this restriction can easily be removed provided the overall dimensions, particularly the local radius of curvature, of the stamp are sufficiently large in comparison with the characteristic dimension of the contact area so that the stamp may be approximated by a half space (see, for example, [3] and [10]).

Another assumption which is generally made in contact problems refers to the interface conditions between the layer and the subspace. It is invariably assumed that along the interface either the displacements and the stresses, or only the normal components of the displacements and the stresses are continuous. The former boundary conditions correspond to the practical problem of an elastic layer which is perfectly bonded to the subspace. However, from the physical viewpoint the latter conditions, which also include

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zero shear stress along the interface, are clearly unrealistic as they imply that tensile as well as compressive tractions can be transmitted through a frictionless contact. Under sufficiently high body forces (such as gravity) generating an additional compressive contact stress, the resulting normal tractions may remain compressive throughout the interface and hence, the second set of interface boundary conditions stated above may be justified. However, this has to be demonstrated in each particular problem under consideration. In the absence of friction between the elastic layer and the subspace the interface boundary conditions which are physically meaningful would be that of "receding contact" [13]. According to this concept, since only compressive contact stress can be transmitted through the frictionless interface, upon the application of the load to the stamp the width of the contact area on the layer-subspace interface becomes finite. As the magnitude of the stamp load is varied, this width may either remain constant or may continuously vary depending on the profile of the stamp.

For concentrated or distributed known normal tractions applied to the free surface of the layer the problem was considered in [14–16]. The layer is assumed to be a "plate" in [14] and an elastic continuum in [15] and [16]. The solution given in [15] is approximate. [16] gives an elegant treatment of the plane as well as the axisymmetric problem. In [16], through the application of standard techniques, the related set of dual integral equations is reduced to a Fredholm type integral equation which turns out to be homogeneous. This implies that conceptually the problem is of an eigenvalue type and hence, for a given system of external loads. The contact area is independent of the magnitude of the loads.

In this paper we will consider the contact problem for an elastic layer in which the external load is applied to the layer through a frictionless rigid stamp. It will be assumed that the contact between the layer and the elastic subspace is frictionless and the interface may transmit only compressive stresses. In this problem the contact stresses under the stamp as well as on the interface are unknown. The problem is formulated in terms of a system of singular integral equations in which the unknown functions are the contact stresses. The formulation offers some obvious advantages, the most important ones being (a) it is quite general and can easily be extended to the problem of multiple stamps, (b) the effect of friction can be taken into account by incorporating the technique followed in [3] into the formulation, (c) the solution directly gives the most important physical quantities, namely the contact stresses, and (d) the type of singular integral equations arising from the formulation and the related methods of solution have been extensively studied.

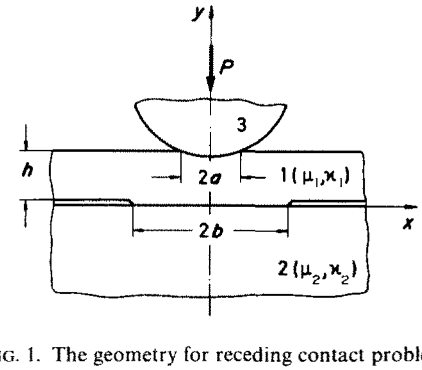

FIG. 1. The geometry for receding contact problem.

2. DERIVATION OF THE INTEGRAL EQUATIONS

Referring to Fig. 1, let an elastic layer 1 lying on an elastic subspace 2 be subjected to a compressive load P through a rigid stamp 3. Let the thickness of the layer be h and the elastic constants of the two materials be μ_i, κ_i, $(i = 1, 2)$ where μ_i is the shear modulus and $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for generalized plane stress, ν_i being the Poisson's ratio. Assuming that $x = 0$ is a plane of symmetry for the stamp and for the external loads, the displacements and the relevant stress components in the layer and in the subspace may be expressed in terms of the following Fourier integrals (see, e.g. [17]):

$$u_1(x, y) = \frac{2}{\pi}\int_0^\infty [(A_1+A_2y)\,e^{-\alpha y}+(A_3+A_4y)\,e^{\alpha y}]\sin \alpha x\,d\alpha,$$

$$v_1(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[A_1+\left(\frac{\kappa_1}{\alpha}+y\right)A_2\right]e^{-\alpha y}+\left[-A_3+\left(\frac{\kappa_1}{\alpha}-y\right)A_4\right]e^{\alpha y}\right\}\cos \alpha x\,d\alpha,$$

$$\frac{1}{2\mu_1}\sigma_{yy}^1 = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_1+A_2y)+\frac{\kappa_1+1}{2}A_2\right]e^{-\alpha y}\right.$$
$$\left.+\left[-\alpha(A_3+A_4y)+\frac{\kappa_1+1}{2}A_4\right]e^{\alpha y}\right\}\cos \alpha x\,d\alpha, \tag{1a–d}$$

$$\frac{1}{2\mu_1}\sigma_{xy}^1 = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_1+A_2y)+\frac{\kappa_1-1}{2}A_2\right]e^{-\alpha y}\right.$$
$$\left.+\left[\alpha(A_3+A_4y)-\frac{\kappa_1-1}{2}A_4\right]e^{\alpha y}\right\}\sin \alpha x\,d\alpha,$$

$$u_2(x, y) = \frac{2}{\pi}\int_0^\infty \{B_1+B_2y\}\,e^{\alpha y}\sin \alpha x\,d\alpha,$$

$$v_2(x, y) = \frac{2}{\pi}\int_0^\infty \left\{-B_1+\left(\frac{\kappa_2}{\alpha}-y\right)B_2\right\}e^{\alpha y}\cos \alpha x\,d\alpha,$$

$$\frac{1}{2\mu_2}\sigma_{yy}^2 = \frac{2}{\pi}\int_0^\infty \left\{-\alpha(B_1+B_2y)+\frac{\kappa_2+1}{2}B_2\right\}e^{\alpha y}\cos \alpha x\,d\alpha, \tag{2a–d}$$

$$\frac{1}{2\mu_2}\sigma_{xy}^2 = \frac{2}{\pi}\int_0^\infty \left\{\alpha(B_1+B_2y)-\frac{\kappa_2-1}{2}B_2\right\}e^{\alpha y}\sin \alpha x\,d\alpha,$$

where the unknowns A_i, $(i = 1,\ldots,4)$ and B_j, $(j = 1, 2)$ are functions of α, which are determined from the boundary conditions at $y = 0$ and $y = h$.

Assuming that the contact between the stamp and the layer and that between the layer and the subspace are frictionless, the boundary conditions may be expressed as†

$$\sigma_{xy}^1 = 0, \qquad \sigma_{xy}^2 = 0, \qquad \sigma_{yy}^1 = \sigma_{yy}^2, \qquad (y = 0, -\infty < x < \infty) \tag{3a–c}$$

† Here the problem is formulated under the physical assumption that the contact between the layer and the subspace is along a single arc, $|x| < b$, or, more precisely, the contact stress is zero for $|x| > b$, b being an unknown. The condition that the contact stress must be compressive for $|x| < b$ has to be separately verified.

$$\left.\begin{aligned} &\sigma_{yy}^2 = 0, \qquad (y = 0, |x| > b), \\ &\frac{\partial}{\partial x}[v_1(x, +0) - v_2(x, -0)] = 0, \qquad (|x| < b), \end{aligned}\right\} \tag{3d}$$

$$\sigma_{xy}^1 = 0, \qquad (y = \mathrm{h}, -\infty < x < \infty), \tag{4a}$$

$$\left.\begin{aligned} &\sigma_{yy}^1 = 0, \qquad (y = h, |x| > a), \\ &\frac{\partial}{\partial x} v_1(x, h) = f(x), \qquad (|x| < a), \end{aligned}\right\} \tag{4b}$$

where $f(x)$ is a known function which is obtained by differentiating the equation giving the profile of the stamp. In (3d) and (4b), the continuity conditions for the displacement are expressed in terms of derivatives for dimensional consistency in the integral equations of the problem.† The constants a and b specifying the contact areas are unknown and are determined from the following equilibrium conditions:

$$\int_{-a}^{a} \sigma_{yy}^1(x, h)\,\mathrm{d}x = -P, \tag{5}$$

$$\int_{-b}^{b} \sigma_{yy}^2(x, 0)\,\mathrm{d}x = -P. \tag{6}$$

Conditions (3a–c) and (4a) give four algebraic equations in A_i and B_j which are used to eliminate four unknowns. The remaining two unknown functions are obtained from the system of dual integral equations given by the mixed conditions (3d) and (4b). Here, instead of writing down the system of dual integral equations, we will first define the following two unknown functions:

$$p_1(x) = \sigma_{yy}^1(x, h), \qquad p_2(x) = \sigma_{yy}^2(x, 0). \tag{7a, b}$$

From (3d) and (4b), it follows that

$$\begin{aligned} p_1(x) &= 0, \qquad (|x| > a), \\ p_2(x) &= 0, \qquad (|x| > b). \end{aligned} \tag{8a, b}$$

It is now clear that if (3d) and (4b) are replaced by (7a, b), substituting from (1) and (2) into (3) and (4) and inverting, all the unknown functions A_i and B_j may be expressed in terms of p_1 and p_2 in the following form:

$$\begin{aligned} A_i(\alpha) &= m_{i1}(\alpha)\int_0^a p_1(t)\cos\alpha t\,\mathrm{d}t + m_{i2}(\alpha)\int_0^b p_2(t)\cos\alpha t\,\mathrm{d}t, \qquad (i = 1, \ldots, 4), \\ B_j(\alpha) &= n_j(\alpha)\int_0^b p_2(t)\cos\alpha t\,\mathrm{d}t, \qquad (j = 1, 2), \end{aligned} \tag{9a, b}$$

where

$$\begin{aligned} n_1(\alpha) &= \frac{\kappa_2 - 1}{4\mu_2\alpha}, \qquad n_2(\alpha) = \frac{1}{2\mu_2}, \\ m_{11}(\alpha) &= \frac{1 - \kappa_1 - 2\alpha h + (\kappa_1 - 1 + 2\kappa_1\alpha h)\,\mathrm{e}^{2\alpha h}}{4\mu_1\alpha\,\mathrm{e}^{\alpha h}(2 + 4\alpha^2h^2 - \mathrm{e}^{-2\alpha h} - \mathrm{e}^{2\alpha h})}, \end{aligned} \tag{10}$$

† For a detailed discussion see [18].

The expressions for the remaining known functions $m_{ik}(\alpha)$, which are omitted, are similar to $m_{11}(\alpha)$ given by (10). The functions A_i and B_j as given by (9) satisfy all the boundary conditions in (3) and (4) except the second part of the mixed conditions (3d) and (4b). Substituting from (9) through (1) and (2) into these latter conditions, we obtain the system of integral equations for the unknown functions p_1 and p_2. Thus, by using the symmetry conditions $p_i(x) = p_i(-x)$, $(i = 1, 2)$ and after separating the singular part of the kernels, we obtain the following system of singular integral equations.†

$$\int_{-a}^{a} \frac{p_1(t)\,dt}{t-x} + \int_{-a}^{a} k_{11}(x,t)p_1(t)\,dt + \int_{-b}^{b} k_{12}(x,t)p_2(t)\,dt = \frac{4\pi\mu_1}{1+\kappa_1} f(x), \qquad (|x| < a),$$

$$\int_{-b}^{b} \frac{p_2(t)\,dt}{t-x} + \int_{-a}^{a} k_{21}(x,t)p_1(t)\,dt + \int_{-b}^{b} k_{22}(x,t)p_2(t)\,dt = 0, \qquad (|x| < b), \tag{11a, b}$$

where

$$k_{11}(x,t) = 2\int_0^\infty \frac{1+2\alpha h+2\alpha^2h^2-e^{-2\alpha h}}{1+e^{-4\alpha h}-(2+4\alpha^2h^2)\,e^{-2\alpha h}}\,e^{-2\alpha h}\sin\alpha(t-x)\,d\alpha$$

$$k_{12}(x,t) = 2\int_0^\infty \frac{-1-\alpha h+(1-\alpha h)\,e^{-2\alpha h}}{1+e^{-4\alpha h}-(2+4\alpha^2h^2)\,e^{-2\alpha h}}\,e^{-\alpha h}\sin\alpha(t-x)\,d\alpha$$

$$k_{21}(x,t) = \frac{1+\beta}{2}k_{12}(x,t) \tag{12a–e}$$

$$k_{22}(x,t) = \frac{1+\beta}{2}k_{11}(x,t)$$

$$\beta = \frac{(1+\kappa_1)\mu_2-(1+\kappa_2)\mu_1}{(1+\kappa_1)\mu_2+(1+\kappa_2)\mu_1}$$

Here, the bi-material constant β is the same as that found in [16]. However, it should be pointed out that if $f(x)$ is not zero, the solution will depend on μ_1 and κ_1 as well as on β.

The infinite integrals giving the Fredholm Kernels $k_{ij}(x,t)$, $(i\ j = 1, 2)$ in (12) may easily be evaluated by using Filon's integration formula [19]. However, if one examines the integrands of k_{ij}, it may be seen that they have a double pole at $\alpha = 0$. Hence, the integrals, if treated separately, will be divergent and their evaluation requires special care. Writing the functions $k_{ij}(x,t)$, $(i, j = 1, 2)$ as

$$k_{ij}(x,t) = \int_0^\infty h_{ij}(x,t,\alpha)\,d\alpha$$

$$= \int_0^\varepsilon h_{ij}(x,t,\alpha)\,d\alpha + \int_\varepsilon^\infty h_{ij}(x,t,\alpha)\,d\alpha \tag{13}$$

† The procedure used in this paper for separating the singular kernels is identical to that followed in [17]. Hence, the intermediate steps leading to the singular integral equations have been omitted.

where ε is an arbitrarily small positive constant, for example the integral equation (11a) may be expressed in the following form:

$$\int_{-a}^{a} \frac{p_1(t)\,dt}{t-x} + \int_{-a}^{a} p_1(t)\,dt \left[\int_0^{\varepsilon} h_{11}(x,t,\alpha)\,d\alpha + \int_{\varepsilon}^{\infty} h_{11}(x,t,\alpha)\,d\alpha \right]$$
$$+ \int_{-b}^{b} p_2(t)\,dt \left[\int_0^{\varepsilon} h_{12}(x,t,\alpha)\,d\alpha + \int_{\varepsilon}^{\infty} h_{12}(x,t,\alpha)\,d\alpha \right] = \frac{4\mu_1\pi}{1+\kappa_1} f(x), \qquad (|x| < a). \tag{14}$$

It is not difficult to show that the integrals evaluated in the range $\varepsilon \leq \alpha \leq \infty$ in (13) and (14) are uniformly convergent and will not cause any trouble. On the other hand, for small values of α, h_{11} and h_{12} may be expressed as

$$h_{11}(x,t,\alpha) = \frac{6h(t-x)\alpha^2 - 12h^2(t-x)\alpha^3 + [14h^3(t-x) - h(t-x)^3]\alpha^4 + 0(\alpha^5)}{h^4\alpha^4 - 2h^5\alpha^5 + 0(\alpha^6)},$$
$$h_{12}(x,t,\alpha) = -\frac{6h(t-x)\alpha^2 - 12h^2(t-x)\alpha^3 + [14h^3(t-x) - h(t-x)^3]\alpha^4 + 0(\alpha^5)}{h^4\alpha^4 - 2h^5\alpha^5 + 0(\alpha^6)}. \tag{15a, b}$$

Clearly, at $\alpha = 0$ h_{ij} has a pole of order 2, which leads to divergent kernels. However, in the integral equation if we consider the terms involving the integrals around $\alpha = 0$ together, using (15), (14) becomes

$$\int_{-a}^{a} \frac{p_1(t)\,dt}{t-x} + \int_{-a}^{a} p_1\,dt \int_{\varepsilon}^{\infty} h_{11}\,d\alpha + \int_{-b}^{b} p_2\,dt \int_{\varepsilon}^{\infty} h_{12}\,d\alpha$$
$$+ \int_{-a}^{a} p_1(t)(t-x)\,dt \int_0^{\varepsilon} \left[\frac{6}{h^3\alpha^2} + 0(1) \right] d\alpha$$
$$- \int_{-b}^{b} p_2(t)(t-x)\,dt \int_0^{\varepsilon} \left[\frac{6}{h^3\alpha^2} + 0(1) \right] d\alpha = \frac{4\pi\mu_1}{1+\kappa_1} f(x), \qquad (-a < x < a), \tag{16}$$

where the terms $0(1)$ contain the variables x and t. Thus, from (16) it is seen that since

$$\int_{-a}^{a} p_1(t)t\,dt = 0, \qquad \int_{-b}^{b} p_2(t)t\,dt = 0,$$
$$\int_{-a}^{a} p_1(t)\,dt = \int_{-b}^{b} p_2(t)\,dt, \tag{17}$$

considered together, the coefficients of the divergent integrals in (15) will be zero. Hence, basically k_{ij} $(i, j = 1, 2)$ may be treated as Fredholm kernels.

In practice it is not necessary to split the range of integration $(0, \infty)$ into $(0, \varepsilon)$ and (ε, ∞) in evaluating the Fredholm kernels. From (15–17) it is seen that around $\alpha = 0$ the first terms in the numerators and the denominators of h_{ij} which will contribute to the kernels k_{ij} are those containing α^4, and because of (17) α^2 and α^3 terms in the numerators will have no contribution. This amounts to saying that one can evaluate the kernels k_{ij} from the integrals given in (12) by simply assuming that the value of the integrand h_{ij}, $(i, j = 1, 2)$ at $\alpha = 0$ is equal to the ratio of the fourth derivative of its numerator to the fourth derivative of its denominator evaluated at $\alpha = 0$, which is finite.

3. A SPECIAL CASE: CONCENTRATED LOADS

First consider the simple case of two concentrated loads applied to the layer at $y = h$, $x = \mp t_0$. This may be considered as the special case in which the stamp degenerates into two disconnected knife edges. Here $p_1(t)$ is known and is given by

$$p_1(t) = -P\delta(t-t_0) - P\delta(t+t_0), \tag{18}$$

where P is the concentrated load per unit length in z-direction. The unknown contact stress $p_2(x)$ between the layer and the subspace must again satisfy (11b) which, using (18), may be written as

$$\int_{-b}^{b} \frac{p_2(t)\,dt}{t-x} + \int_{-b}^{b} k_{22}(x,t)p_2(t)\,dt = P[k_{21}(x,t_0) + k_{21}(x,-t_0)], \qquad (|x| < b). \tag{19}$$

In (19) the contact width $2b$ is unknown which is determined from the following equilibrium condition:

$$\int_{-b}^{b} p_2(x)\,dx = -2P. \tag{20}$$

Dividing both sides by P and referring to [20], from (19) and (20) it is immediately seen that the unknown function p_2/P and the constant b can uniquely be determined from the solution of the singular integral equations. It then follows that the constant b will depend on the location t_0 of the load but will be independent of its magnitude P. From (19) it is also seen that if P is replaced by distributed loads, b again will be independent of the magnitude of the load which is a multiplicative constant in the expression of the external loads. This is the same conclusion arrived at in [14–16].

To solve (19), it is convenient to define the following dimensionless variables and function:

$$\begin{gathered} r = x/b, \qquad s = t/b, \qquad \lambda = \alpha b, \qquad H = h/b \\ s_0 = t_0/b, \qquad \varphi(r) = p_2(rb)/P. \end{gathered} \tag{21}$$

With (21), (19) and (20) may be written as

$$\int_{-1}^{1} \frac{\varphi(s)}{s-r}\,ds + \int_{-1}^{1} K_{22}(H,r,s)\varphi(s)\,ds = K_{21}(H,r,s_0) + K_{21}(H,r,-s_0), \qquad (|r| < 1), \tag{22}$$

$$b\int_{-1}^{1} \varphi(s)\,ds = -2, \tag{23}$$

where $K_{ij}(H,r,s)$ is obtained from k_{ij} by simply replacing h, x, t by H, r, s, respectively. For a given set of constants t_0 and h even though (22) and (23) has a unique solution, it can only be determined by some kind of interpolation. A simple procedure to evaluate b and $\varphi(r)$ would be the following: For a given $s_0 = t_0/b$ solve (22) and compute

$$\Phi(b) = b\int_{-1}^{1} \varphi(s)\,ds \tag{24}$$

for various values of $H = h/b$. The correct value of b is that satisfying $\Phi(b) = -2$, which can be obtained by interpolation to any desired degree of accuracy. The integral equation

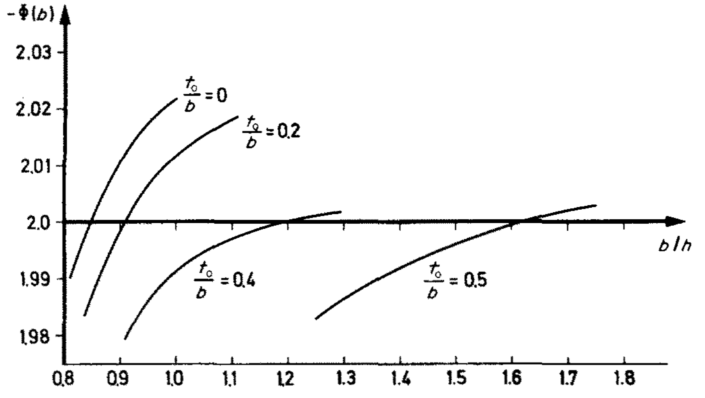

FIG. 2. The variation of $\phi(b)$ with b/h for various values of t_0, ($\beta = 1{\cdot}0$).

is solved by using the technique described in [21]. Figure 2 shows some sample results giving $\Phi(b)$ as a function of b/h for various values of t_0/b and $\beta = 1$.

Figures 3–5 show the calculated results for the concentrated loads. Figure 3 shows the contact width b as a function of β for $-1 < \beta < 1$ corresponding to $\infty > (\mu_1/\mu_2) > 0$. It is seen that as $(\mu_1/\mu_2) \to \infty$ $b \to \infty$. Figure 4 shows the distribution of the contact stress p_2 for $t_0 = 0$ and for various values of β. Figure 5 shows the function p_2 for various values of t_0 calculated from $\beta = 1$ which corresponds to the case of a rigid subspace. At approximately $t_0/h = 1{\cdot}2$ it is seen that the contact stress at $x = 0$ becomes zero. Therefore for $t_0/h > 1{\cdot}2$ there will be two contact areas along $-b_2 < x < -b_1$ and $b_1 < x < b_2$. In this case since the general solution (1) and (2) is given for $x > 0$, it is possible (and preferable) to express the integral equation for $x > 0$ and $t > 0$. Thus the singular integral equation again involves a single arc $b_1 < x < b_2$ and its solution may be obtained by using the technique described in [21]. In this double contact problem the two conditions

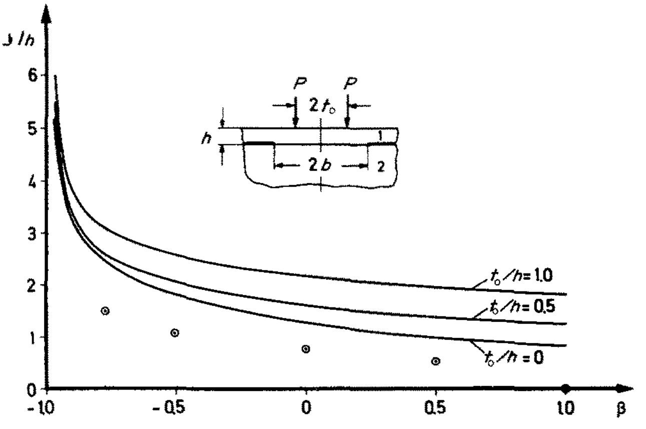

FIG. 3. The variation of the contact width with β. The circles correspond to the result given in [22].

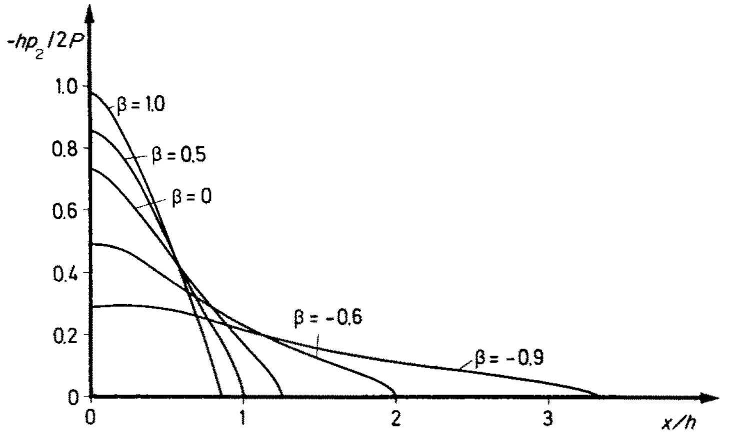

FIG. 4. The distribution of contact stress for various values of β, ($t_0 = 0$).

giving the constants b_1, and b_2 are

$$\int_{b_1}^{b_2} p_2(x)\,\mathrm{d}x = -P \tag{25}$$

and, noting that the index is -1, the consistency condition of the integral equation.

4. SOLUTION FOR A FLAT STAMP

Let the layer be loaded by a rigid stamp with a flat profile (see the insert in Fig. 6). The system of integral equations (11) must now be solved under

$$f(x) = 0, \qquad \int_{-a}^{a} p_1(t)\,\mathrm{d}t = -P, \qquad \int_{-b}^{b} p_2(t)\,\mathrm{d}t = -P. \tag{26a–c}$$

Dividing both sides of (11a, b) and (26b, c) by P, we obtain a system of homogeneous singular integral equations for the unknown functions p_1/P and p_2/P which must be

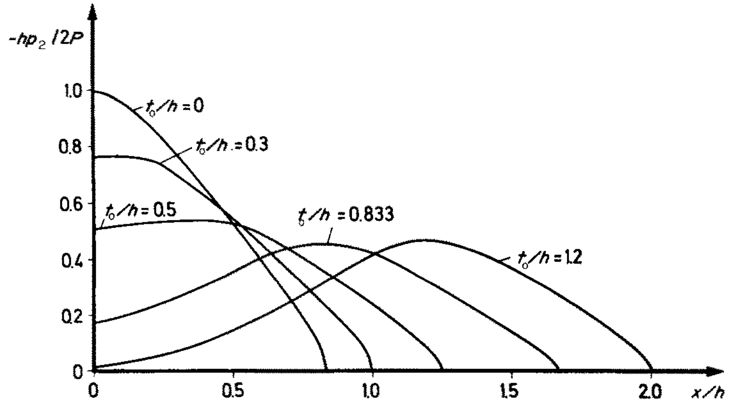

FIG. 5. The distribution of contact stress for various values of t_0/h, ($\beta = 1{\cdot}0$).

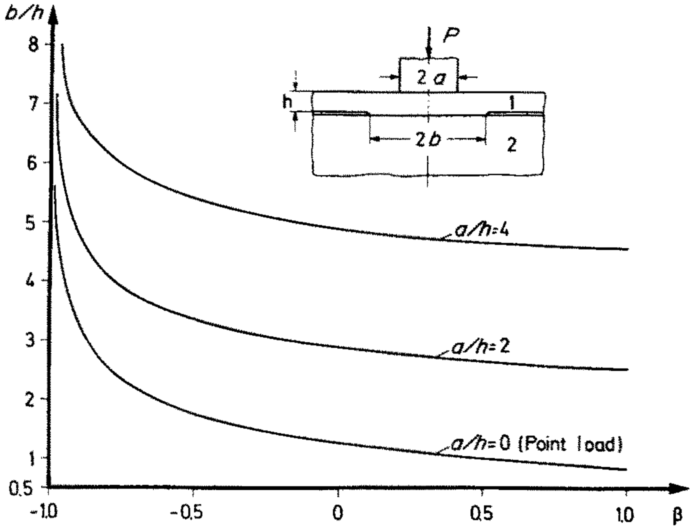

FIG. 6. The variation of contact width with β for loading by a flat stamp.

solved under a set of nonhomogeneous conditions (26b, c). At the end points $x = \mp a$ the function p_1 has integrable singularities, whereas at $x = \mp b$, p_2 is bounded (and is zero). Thus the index of (11a) is $+1$ and its general solution will contain an arbitrary constant (see [20]). (26b) is the additional condition which accounts for this constant. On the other hand the index of (11b) is -1, which also contains the unknown constant b. Theoretically, (26c) is the condition which accounts for this constant. It is then clear that (11) and (26) will provide a unique solution for the unknowns $p_1(x)/P$, $p_2(x)/P$, and b. This means that in this problem too the width of the contact area b will be independent of the magnitude of the external load P.

To solve the integral equations (11) we again define a set of dimensionless variables and functions. Note that (11a) and (11b) are the displacement continuity relations at $y = h$ and $y = 0$, respectively. Hence, by designating the variables (x, t) on $y = h$ and $y = 0$ by (x_1, t_1) and (x_2, t_2), respectively we may now define

$$\begin{gathered} r_1 = x_1/a, \qquad s_1 = t_1/a, \qquad \lambda = \alpha b, \qquad H = h/b \\ r_2 = x_2/b, \qquad s_2 = t_2/b, \\ p_1(as_1)/P = \varphi_1(s_1), \qquad p_2(bs_2)/P = \varphi_2(s_2). \end{gathered} \tag{27}$$

Substituting from (27), the integral equations (11) and the equilibrium conditions (26b, c) may be expressed as

$$\int_{-1}^{1} \frac{\varphi_i(s_i)}{s_i - r_i} \mathrm{d}s_i + \int_{-1}^{1} \sum_{j=1}^{2} M_{ij}(r_i, s_j)\varphi_j(s_j)\,\mathrm{d}s_j = 0, \qquad (i = 1, 2, |r_i| < 1), \tag{28a, b}$$

$$a\int_{-1}^{1} \varphi_1(s_1)\,\mathrm{d}s_1 = -1, \qquad b\int_{-1}^{1} \varphi_2(s_2)\,\mathrm{d}s_2 = -1, \tag{29a, b}$$

where the kernels M_{ij} are given by

$$M_{11}(r_1, s_1) = \frac{a}{b}\int_0^\infty F_{11}(\lambda H)\sin\left[\frac{a}{b}\lambda(s_1 - r_1)\right] \mathrm{d}\lambda,$$

$$M_{12}(r_1, r_2) = \int_0^\infty F_{12}(\lambda H)\sin\left[\lambda\left(s_2 - \frac{a}{b}r_1\right)\right] \mathrm{d}\lambda,$$

$$M_{21}(r_2, s_1) = \frac{a}{b}\int_0^\infty F_{21}(\lambda H)\sin\left[\lambda\left(\frac{a}{b}s_1 - r_2\right)\right] \mathrm{d}\lambda,$$

$$M_{22}(r_2, s_2) = \int_0^\infty F_{22}(\lambda H)\sin[\lambda(s_2 - r_2)]\,\mathrm{d}\lambda;$$

$$F_{11}(\lambda H) = 2\,\mathrm{e}^{-2\lambda H}\frac{1 + 2\lambda H + 2\lambda^2 H^2 - \mathrm{e}^{-2\lambda H}}{1 + \mathrm{e}^{-4\lambda H} - (2 + 4\lambda^2 H^2)\,\mathrm{e}^{-2\lambda H}},$$

$$F_{12}(\lambda H) = 2\,\mathrm{e}^{-\lambda H}\frac{-1 - \lambda H + (1 - \lambda H)\,\mathrm{e}^{-2\lambda H}}{1 + \mathrm{e}^{-4\lambda H} - (2 + 4\lambda^2 H^2)\,\mathrm{e}^{-2\lambda H}},$$

$$F_{21}(\lambda H) = \frac{1+\beta}{2}F_{12}(\lambda H),$$

$$F_{22}(\lambda H) = \frac{1+\beta}{2}F_{11}(\lambda H). \tag{30a–h}$$

Since variables r_i and s_j all have the same range $(-1, 1)$, in the subsequent analysis it is not necessary to keep the indexes $(i, j = 1, 2)$ in (28–30).

The fundamental functions of the singular integral equations (28a) and (28b), respectively, are (see [20])

$$w_1(s) = (1 - s^2)^{-\frac{1}{2}}, \qquad w_2(s) = (1 - s^2)^{\frac{1}{2}}. \tag{31a, b}$$

Thus, noting that w_1 and w_2 are, respectively, the weight functions of the Chebyshev polynomials $T_n(s)$ and $U_n(s)$ and taking into account the symmetry properties $\varphi_i(s) = \varphi_i(-s)$, $(i = 1, 2)$, the solution of (28) may be expressed as

$$\varphi_1(s) = w_1(s)\sum_0^\infty A_n T_{2n}(s), \qquad \varphi_2(s) = w_2(s)\sum_0^\infty B_n U_{2n}(s). \tag{32a, b}$$

The unknown constants A_n and B_n are obtained (in terms of b) by substituting from (32) into (28a, b) and (29a) and by using the method described in [21]. To determine b we again define

$$\Phi(b) = b\int_{-1}^{1} \varphi_2(s)\,\mathrm{d}s. \tag{33}$$

From (29b) it is seen that for the correct value of b we have $\Phi(b) = -1$. Thus the correct value of b (as well as that of φ_1 and φ_2) may be obtained by solving the integral equations for various values of b and interpolating in $\Phi(b)$ vs. b plane (see, e.g., Fig. 2).

Figures 6 and 7 show some calculated results. Figure 6 shows the variation of the half width of the contact area between the layer and the half space as a function of the bielastic constant β which is given by (12e). The results are given for $a/h = 4$, 2 and 0

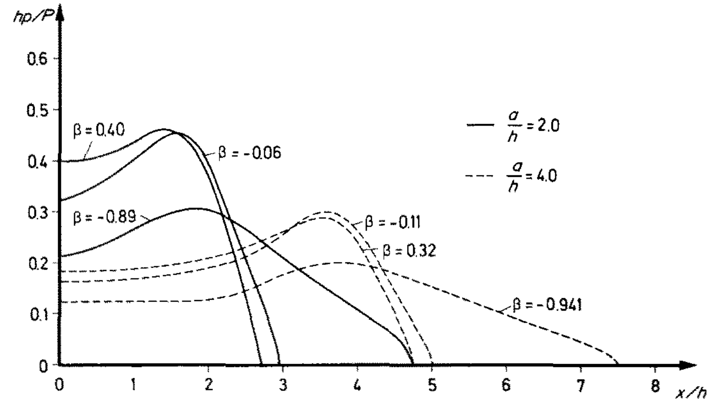

FIG. 7. The pressure distribution between the layer and the half space for various values of β and for loading by a flat stamp.

where a is the half width of the stamp, h is the layer thickness and the last case, i.e., $a = 0$ corresponds to the concentrated load which was discussed in the previous section. It is again seen that as $\mu_2/\mu_1 \to 0$, i.e. as $\beta \to -1$, $b \to \infty$.

The distribution of the contact pressure between the layer and the subspace is shown in Fig. 7. Here the two sets of curves plotted for various values of β again correspond to $a/h = 2$ and $a/h = 4$. Note that the contact pressure has peaks in the neighborhood of the stamp ends $x = \mp a$.

5. SOLUTION FOR A CURVED STAMP

Consider now the problem for a symmetric curved rigid stamp with a local radius of curvature R (see the insert in Fig. 8). In this case the contact width $2a$ at $y = h$ as well as the contact width $2b$ at $y = 0$ is unknown, and the input function $\partial v_1(x, h)/\partial x$ is given by†

$$f(x) = x/R, \qquad (|x| < a). \tag{34}$$

The unknown functions p_1 and p_2 are bounded at the end points $x = \mp a$ and $x = \mp b$, respectively. Hence, the index for both singular integral equations (11a) and (11b) is -1, and consequently the solution of the nonhomogeneous system (11) does not involve any arbitrary constants [20]. The unknown constants a and b can then be determined from the equilibrium conditions (26b, c). It is now clear that in this problem the contact width $2b$ between the layer and the subspace will not be independent of the magnitude of the external load P

To solve the problem, in addition to the dimensionless variables defined by (27), we define the following functions:

$$p_1(as_1) = \psi_1(s_1), \qquad p_2(bs_2) = \psi_2(s_2). \tag{35a, b}$$

† Within the confines of other assumptions inherent in the linear theory of elasticity, the solution given in this section is valid for any value of a/R if the profile of the stamp is a parabola given by $y = x^2/2R$ + constant. Otherwise it is assumed that the depth of penetration and the contact width are "small" compared to the local radius of curvature R.

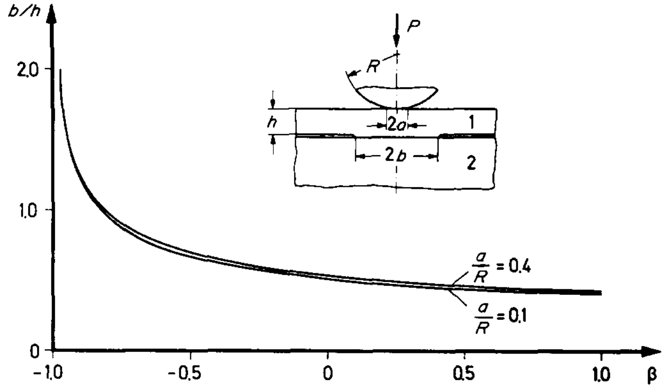

FIG. 8. The variation of the layer-subspace contact width with β for loading by a curved stamp, $(R = h)$.

The integral equations (11) and the equilibrium conditions (26b, c) may then be expressed as

$$\int_{-1}^{1} \frac{\psi_i(s)}{s-r} \mathrm{d}s + \int_{-1}^{1} \sum_{1}^{2} M_{ij}(r, s)\psi_j(s) \, \mathrm{d}s = g_i(r), \qquad (|r| < 1, i = 1, 2),$$

$$g_1(r) = \frac{4\pi\mu_1}{1+\kappa_1} \frac{r}{R/a}, \qquad g_2(r) = 0, \tag{36a–c}$$

$$a \int_{-1}^{1} \psi_1(s) \, \mathrm{d}s = -P, \qquad b \int_{-1}^{1} \psi_2(s) \, \mathrm{d}s = -P, \tag{37a, b}$$

where the subscripts in the variables r and s are dropped, and the Fredholm kernels M_{ij} are those defined by (30). The fundamental function for both integral equations given by (36) is

$$w(r) = (1-r^2)^{\frac{1}{2}}. \tag{38}$$

Thus the solution of (36) may be expressed as

$$\psi_i(r) = (1-r^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_{in} U_{2n}(r), \qquad (i = 1, 2), \tag{39}$$

where the unknown coefficients A_{in} are again determined by using the technique described in [21]. In the problem the constants κ_1, μ_1, β, h, R and P are given and a, b, $\psi_1(r)$ and $\psi_2(r)$ are unknown. However, in practice the problem is again solved in an inverse manner; namely, it is assumed that instead of the load P, the contact width $2a$ is specified, and ignoring the condition (37b), the integral equations (36a) are solved for various values of b. Then computing the quantity

$$F(b) = b \int_{-1}^{1} \psi_2(s) \, \mathrm{d}s - a \int_{-1}^{1} \psi_1(s) \, \mathrm{d}s, \tag{40}$$

the correct value of b corresponding to the specified value of a is obtained by interpolation in such a way that $F(b) = 0$. After determining b, the load P is obtained from (37).

Figures 8–11 show some of the numerical results. In all the numerical examples considered in this section it was assumed that $R = h$. The variation of the half-contact width,

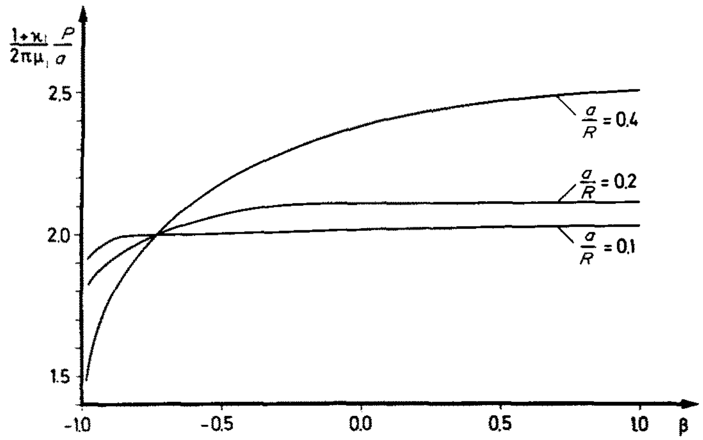

FIG. 9. The variation of P with β for various a/R ratios, $(R = h)$.

b on the layer-subspace interface as a function of the bielastic constant β is shown in Fig. 8 for two (relatively speaking, widely different) values of $a/R = 0{\cdot}4$ and $a/R = 0{\cdot}1$. The results obtained for $a/R = 0{\cdot}2$ (which are not shown in the figures) were hardly distinguishable from that corresponding to $a/R = 0{\cdot}1$. The figure shows that for $R = h$ the contact width $2b$ is not appreciably affected by the ratio a/R. However, it should be noted that P varies as β is varied for a fixed value of a/R, and as a/R is varied for a fixed value of β. This may be seen from Fig. 9 where, for various values of a/R, the variation of P is shown as a function of β. From Figs. 8 and 9 it may then be concluded that in the case of a curved stamp for given dimensions h and R and a given material constant β, unlike the results found in the two previous sections, the contact width $2b$ is a function of the magnitude of the external load P.

Figures 10 and 11 show the distribution of the contact stress between the layer and the subspace for $a/R = 0{\cdot}1$ and $a/R = 0{\cdot}4$, respectively. The values of P corresponding to various values of β used in these figures are different and may be obtained from Fig. 9.

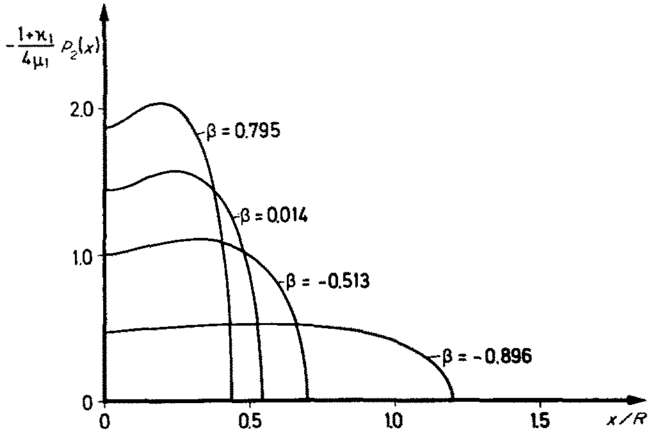

FIG. 10. The distribution of the contact stress between the layer and the subspace, $(a/R = 0{\cdot}1, h = R)$.

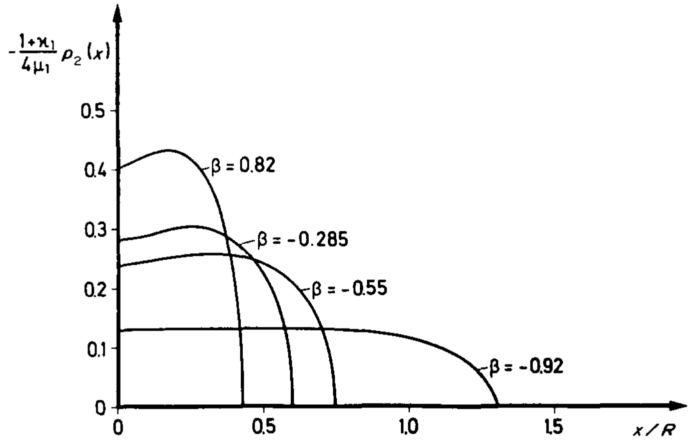

FIG. 11. The distribution of the contact stress between the layer and the subspace, (a/R = 0·4, $h = R$).

Perhaps a somewhat unexpected feature of the pressure distributions shown in these figures is that their peak generally is not at $x = 0$.

6. DISCUSSION AND CONCLUDING REMARKS

The result for the concentrated load, i.e. $t_0 = 0$, shown in Fig. 3 is indistinguishable from that given in [16], where an entirely different method was used to solve the contact problem. Comparison of the Figs. 3 and 8 indicates that the result for the curved stamp with $(a/R) = 0{\cdot}1$, and $R = h$ is very nearly the same as that of the concentrated load. This may also be seen from Fig. 9 where for $(a/R) = 0{\cdot}1$ the resultant load P is nearly independent of β.

The concentrated load result for the problem under consideration using the classical plate (or beam) theory† was recently given in [22], where the following expression for the contact width was obtained:

$$\frac{b}{h} = 0{\cdot}737\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{3}} \tag{41}$$

The values obtained from (41) are shown in Fig. 3 as small circles. The agreement appears to be rather poor. The discrepancy may be due to the highly approximate nature of the theory as well as the method of solution of the related integral equation employed in [22].

By using the technique described in this paper, the problems of a nonsymmetric stamp and multiple stamps may be treated without too much difficulty. In principle all these problems may be reduced to a system of singular integral equations defined on a set of nonintersecting arcs the theory of which has been extensively studied [20].

Since the symmetric problem for a flat stamp with sharp corners discussed in Section 4 is the only contact problem for which the resulting system of singular integral equations

† The equations used in [22] for the half-space are that of plane strain and are appropriate for the plate problem. However, for the beam problem the plane stress modification of these equations would have been physically more meaningful.

is homogeneous, only for this case the width of the contact area between the layer and the subspace will be independent of the magnitude of the compressive load applied to the stamp. For all other stamp geometries including those for which the stamp width $2a$ is constant, the contact width on the layer–subspace interface will be dependent on the magnitude of the applied load.

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Абстракт—Обсуждается плоская контактная задача для упругого слоя, лежащего на упругом полупространстве. Сжимаемая нагрузка приложена к слою посредством жесткого штампа, без учета трения. Подразумевается, что нет трения в конуакте слоя с подпространством и только сжимаемые нормальные тяговые усилия могут передаваться сквозь поверхность раздела. Следовательно, ширина области соприкосновения по поверхности между слоем и подпространством конечна и неизвестна. Формулируется задача в виде системы сингулярных интегральный уравнений, неизвестные функции которых являются давлением между штампом и слоем и между слоем и подпространством. Впервые, решается задача для двух специальных случаев сосредоточенной нагрузки. Исследуются, затем, два типа геометрии штампа, именно плоский штамп с острыми углами и искривленный штамп. Для случая плоского штампа система интегральных уравнений однородна. Вследствие этого, ширина контакта между слоем и подпространством не зависит от величины внешней нагрузки, приложенной к штампу. Однако, для задачи искривленного штампа размер области района контакта является функцией величины внешней нагрузки.